Name of the Course: Linear Algebra

Syllabus:

Unit I

Symmetric and Skew symmetric matrices - Hermitian and Skew Hermitian Matrices – Orthogonal and Unitary matrices - Rank of matrix - Eigen values and Eigen vectors of Linear operators – Cayley Hamilton theorem - Solutions of Homogeneous linear equations – Solutions of non homogenuous linear equations.

Section 1.2: Hermitian and Skew Hermitian Matrices

Definition 1.2.1: Conjugate Matrix

Let $A = (a_{ij})$ be any given matrix. The conjugate of the complex matrix is defined by $\bar{A} = \overline{(a_{ij})}$

Example: Let $A = \begin{pmatrix} 1 & -2i \\ 2+3i & -1 \end{pmatrix}$. Then $\bar{A} = \overline{\begin{pmatrix} 1 & -2i \\ 2+3i & -1 \end{pmatrix}} = \begin{pmatrix} 1 & 2i \\ 2-3i & -1 \end{pmatrix}$

Definition 1.2.1: Conjugate Transpose of a Matrix

Let $A = (a_{ij})$ be any given matrix. The conjugate transpose of the complex matrix is defined by $A^* = (\bar{A})^T = \overline{(a_{ij})}^T$ or $A^* = \overline{(A^T)}$

Example: Let $A = \begin{pmatrix} 1 & -2i \\ 2+3i & -1 \end{pmatrix}$. Then $\bar{A} = \overline{\begin{pmatrix} 1 & -2i \\ 2+3i & -1 \end{pmatrix}} = \begin{pmatrix} 1 & 2i \\ 2-3i & -1 \end{pmatrix}$

$A^* = \begin{pmatrix} 1 & 2i \\ 2-3i & -1 \end{pmatrix}^T = \begin{pmatrix} 1 & 2-3i \\ -2i & -1 \end{pmatrix}$

Theorem 1.2.2

Let A and B are Complex matrices. Then their product AB is defined.

(i) $\overline{AB} = \bar{A}.\bar{B}$ (ii) $(AB)^* = B^* A^*$ (iii) $(\bar{A})^{-1} = \overline{(A^{-1})}$ (iv) $(A^*)^{-1} = (A^{-1})^*$

Proof:

(i) $\overline{AB} = \overline{a_{i1}}$

Let $A = (a_{rs})_{m\times n}$ and $B = (b_{kl})_{n\times p}$. Then their product AB is defined. (i,j) th entry of $\overline{AB}$ is

$$\overline{AB} = \overline{a_{\iota 1}b_{1\jmath} + \cdots\ a_{\iota n}b_{n\jmath}}$$
$$= \overline{a_{\iota 1}b_{1\jmath}} + \overline{a_{\iota 2}b_{2\jmath}} + \ \ldots . + \overline{a_{\iota n}b_{n\jmath}}$$
$$= \overline{a_{\iota 1}}\,\overline{b_{1\jmath}} + \overline{a_{\iota 2}}\,\overline{b_{2\jmath}} + \ \ldots . + \overline{a_{\iota n}}\,\overline{b_{n\jmath}}$$
$= \text{(i,j) th entry of } \bar{A}\,.\bar{B}$

$$\overline{=\ A}.\bar{B}$$

(ii) $(AB)^* = (\bar{A}.\bar{B})^T$

$$= \bar{B}\ ^T.\ \bar{A}\ ^T$$
$$= B^*\ A^*$$

(iii) W.k.t $I = \bar{I} = \overline{AA^{-1}} = \bar{A}\,\overline{A^{-1}}$

$$I = \bar{A}\,\overline{A^{-1}}$$

Implies that $(\bar{A})^{-1} = \overline{(A^{-1})}$

(iv) $(A^*)^{-1} = ((\bar{A})^T)^{-1}$

$$= \overline{(A^T)^{-1}}$$
$$= \overline{(A^{-1})^T}$$
$$= (A^{-1})^*$$

Problem 1.2.3 : Prove that $(iA)^* = -iA^*$

Solution:

Let $A = (a_{rs})$

Then $(iA)^* = (ia_{rs})^*$

$$= (\overline{\imath a_{rs}})^T$$

$$= -i\ (\overline{a_{sr}})$$

$$= -iA^*$$

Hence the result .

Definition 1.2.4: Hermitian Matrix

A Complex Matrix $H = (h_{ij})$ is called Hermitian Matrix if it satisfies the condition if $H = H^*$. $\Rightarrow h_{ij} = \overline{h_{\jmath\iota}}$ for every i and j.

Definition 1.2.5: Skew Hermitian Matrix

A Complex Matrix $S = (s_{ij})$ is called skew - Hermitian Matrix if it satisfies the condition if $S = -S^*$. $\Rightarrow s_{ij} = -\overline{s_{\jmath\iota}}$ for every i and j.

Problem 1.2.3 : Check whether the following matrix is Hermitian matrix.

$$H = \begin{pmatrix} 1 & 2+3i & -1 \\ 2-3i & \sqrt{5} & \sqrt{2}+i \\ -1 & \sqrt{2}-i & 3/2 \end{pmatrix}$$

Solution: Given that $H = \begin{pmatrix} 1 & 2+3i & -1 \\ 2-3i & \sqrt{5} & \sqrt{2}+i \\ -1 & \sqrt{2}-i & 3/2 \end{pmatrix}$

$$H^T = \begin{pmatrix} 1 & 2-3i & -1 \\ 2+3i & \sqrt{5} & \sqrt{2}-i \\ -1 & \sqrt{2}+i & 3/2 \end{pmatrix}$$

$$\overline{H^T} = \begin{pmatrix} 1 & 2+3i & -1 \\ 2-3i & \sqrt{5} & \sqrt{2}+i \\ -1 & \sqrt{2}-i & 3/2 \end{pmatrix}$$

$$\overline{H^T} = H.$$